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Solutions for Fredholm equations through nonlinear iterative processes

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Abstract. By adjunction of one or two well chosen functionals taking account simultaneously of the unknown $u(x)$ and of the inhomogeneous term $f(x)$, every Fredholm equation of the second kind can easily be transformed into new nonlinear integral equations, for which the solution of the primary equation remains obviously valid. However, when usual iterative solving methods are tested, the convergence of the various sequences now available are very different and we need criteria to select the best ones. Fredholm equations of the first kind can also be solved, using the new processes described which are particularly efficient after a preliminary transform pointing out the first iterated operator which is necessarily positive. Optimisation techniques are detailed, in order to work out nonlinear equations of particular interest, i.e. they are very suitable to perform numerically an accurate iterative solution.

1. Introduction

It is often easy to set up Fredholm integral equations which are related to various physical systems (Lonseth 1977). For instance, in molecular physics, we may usually associate a Fredholm equation of the second kind to each perturbation problem (Guy *et al* 1979b); equations of the first kind, in many cases with a convolution kernel, are also important as in medical imaging (Barrett and Swindell 1977, Guy *et al* 1979a). However, integral operators are rarely utilised in numerous physical areas, as the equations involved are frequently judged too difficult to solve. In order to improve this situation, we have worked out some new and relatively simple iterative methods with a rather good numerical efficiency.

2. Basic principles for solving equations of the second kind

In the following Fredholm equation of the second kind

$$u(x)=f(x)+(\boldsymbol{K}u)(x)=f(x)+\int_D K(x,y)u(y)\,\mathrm{d}y \qquad (f\in L_2, K(x,y)\in L_2), \quad (2.1)$$

where x and y are eventually multidimensional variables, the simultaneous knowledge of the kernel $K(x, y)$ and of the integration domain D defines completely the linear

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operator $\boldsymbol{K}$, the norm of which is explicitly supposed finite. Then, when unity is not an eigenvalue of $\boldsymbol{K}$, equation (2.1) has one and only one solution which can be written as

$$u(x) = f(x) + (\boldsymbol{R}f)(x) = f(x) + \int_D R(x, y) f(y)\, \mathrm{d}y, \tag{2.2}$$

$\boldsymbol{R}$ being the resolvent operator associated to $\boldsymbol{K}$ (Mathews and Walker 1964, Zabreyko *et al* 1975).

As the practical determination of the resolvent kernel $R(x, y)$ is usually difficult, the iterative Liouville–Neumann sequence is often very helpful and consequently used to compute $u(x)$. We have the successive approximations (Margenau and Murphy 1949, Zabreyko *et al* 1975)

$$\begin{aligned} &u_0(x) = f(x), \\ &u_1(x) = f(x) + (\boldsymbol{K}u_0)(x), \\ &\ldots \\ &u_{n+1}(x) = f(x) + (\boldsymbol{K}u_n)(x), \end{aligned} \tag{2.3}$$

which are equivalent to

$$\boldsymbol{R} = \sum_{n=1}^{\infty} \boldsymbol{K}^n. \tag{2.4}$$

However, for any given function $f(x)$, the convergence of the sequence (2.3) will be ensured only when the spectral norm of operator $\boldsymbol{K}$ is strictly less than one. With

$$\|u\| = \left(\int_D [u(x)]^2 \, \mathrm{d}x \right)^{1/2}, \tag{2.5}$$

the norm of $\boldsymbol{K}$ will be given by

$$\|\boldsymbol{K}\| = \left[\sup_{\|u\|=1} \left(\int_D ((\boldsymbol{K}u)(x))^2 \, \mathrm{d}x \right) \right]^{1/2}. \tag{2.6}$$

For $\|\boldsymbol{K}\| \geqslant 1$, iteration (2.3) will not be successful, except when both the kernel and the inhomogeneous term present some remarkable symmetry features (Guy and Mangeot 1981).

3. Nonlinear integral equations related to any Fredholm equation of the second kind

Starting from (2.1), it is not difficult to build up many new and nonlinear related equations, such as

$$u(x) = \Phi_1(u, f) f(x) + \Phi_2(u, f)(\boldsymbol{K}u)(x). \tag{3.1}$$

In (3.1), Φ_1 and Φ_2 are arbitrary functionals with an important property: they become equal to one when $u(x)$ is precisely the solution of (2.1). There is a large choice for

these functionals and we give a few typical examples:

$$\begin{aligned}
&\Phi(u,f)=[\|\boldsymbol{K}u\|/\|u-f\|]^p \qquad (p\neq 0),\\
&\Phi(u,f)=(\boldsymbol{K}f,(\boldsymbol{I}-\boldsymbol{K})\boldsymbol{K}u)/\|(\boldsymbol{I}-\boldsymbol{K})\boldsymbol{K}u\|^2,\\
&\Phi(u,f)=\exp[-\alpha\|u-f-\boldsymbol{K}u\|] \qquad (\alpha>0).
\end{aligned} \tag{3.2}$$

The reader will easily verify that every formula in (3.2) agrees with $\Phi(f+\boldsymbol{K}u,f)=1$.

4. Iterative methods with convergence improvements

Formally, we may try to solve equations (3.1) by using a tentative iteration technique. After a choice of the initial guess $u_0(x)$ (most often, $u_0(x)$ will be either the null function or $f(x)$), the iterative steps are

$$u_{n+1}(x)=\alpha_n f(x)+\beta_n(\boldsymbol{K}u_n)(x), \tag{4.1}$$

where coefficients α_n and β_n have the values

$$\alpha_n=\Phi_1(u_n,f), \qquad \beta_n=\Phi_2(u_n,f). \tag{4.2}$$

The change of (3.1) into (4.1) is very similar to the change of (2.1) into (2.3), the only striking difference being the loss of the linear properties, due to the introduction of both functionals Φ_1 and Φ_2. An empirical comparison between (4.1) and (2.3) shows clearly that the rules of convergence are greatly modified; furthermore, in the nonlinear case (4.1), we cannot exhibit a resolvent operator resembling the geometrical series (2.4).

In order to create a class of Φ-functionals giving as efficient a convergence as possible for (4.1), we have to use suitable criteria, i.e. those which are able to induce such a property; these main ideas are pointed out in Guy *et al* (1965) and Salès *et al* (1968). We may decide, for instance, that $u_{n+1}(x)$ has to satisfy the variational condition

$$\delta\|u_{n+1}-f-\boldsymbol{K}u_{n+1}\|^2=0; \tag{4.3}$$

the analytical form of the functionals Φ_1 and Φ_2 are then a logical consequence of (4.3) and we thus obtain the values of α_n and β_n.

When β_n is the only parameter to be adjusted (i.e. $\forall n$, $\alpha_n=1$), we have

$$\|u_{n+1}-f-\boldsymbol{K}u_{n+1}\|^2=\beta_n^2\|(\boldsymbol{I}-\boldsymbol{K})\boldsymbol{K}u_n\|^2-2\beta_n(\boldsymbol{K}f,(\boldsymbol{I}-\boldsymbol{K})\boldsymbol{K}u_n)+\|\boldsymbol{K}f\|^2 \tag{4.4}$$

and condition (4.3) gives

$$\beta_n=(\boldsymbol{K}f,(\boldsymbol{I}-\boldsymbol{K})\boldsymbol{K}u_n)/\|(\boldsymbol{I}-\boldsymbol{K})\boldsymbol{K}u_n\|^2 \tag{4.5}$$

which minimises (4.4). More explicitly, our method here consists of using a standard iterative process to solve the nonlinear equation

$$u(x)=f(x)+[(\boldsymbol{K}f,(\boldsymbol{I}-\boldsymbol{K})\boldsymbol{K}u)/\|(\boldsymbol{I}-\boldsymbol{K})\boldsymbol{K}u\|^2](\boldsymbol{K}u)(x); \tag{4.6}$$

if $u(x)$ verifies exactly (2.1), $(\boldsymbol{K}u)(x)$ in (4.6) is necessarily multiplied by unity, as

$$(\boldsymbol{K}f,(\boldsymbol{I}-\boldsymbol{K})\boldsymbol{K}u)=\|(\boldsymbol{I}-\boldsymbol{K})\boldsymbol{K}u\|^2=\|\boldsymbol{K}f\|^2. \tag{4.7}$$

In the general case, when α_n and β_n are both simultaneously deduced from (4.3), we obtain

$$\|u_{n+1}-f-\boldsymbol{K}u_{n+1}\|^2 = A^2\alpha_n^2 + B_n^2\beta_n^2 + 2D_n\alpha_n\beta_n - 2F\alpha_n - 2G_n\beta_n + \|f\|^2, \tag{4.8}$$

with the following simplified symbolism

$$\begin{aligned} &A^2 = \|(\boldsymbol{I}-\boldsymbol{K})f\|^2, && B_n^2 = \|(\boldsymbol{I}-\boldsymbol{K})\boldsymbol{K}u_n\|^2, \\ &D_n = ((\boldsymbol{I}-\boldsymbol{K})f, (\boldsymbol{I}-\boldsymbol{K})\boldsymbol{K}u_n), && F = (f, (\boldsymbol{I}-\boldsymbol{K})f), \\ &G_n = (f, (\boldsymbol{I}-\boldsymbol{K})\boldsymbol{K}u_n). \end{aligned} \tag{4.9}$$

Solving (4.3) gives, for the adjusted parameters of (4.1),

$$\alpha_n = \frac{B_n^2F - D_nG_n}{A^2B_n^2 - D_n^2}, \qquad \beta_n = \frac{A^2G_n^2 - D_nF}{A^2B_n^2 - D_n^2}. \tag{4.10}$$

When the iterative process (4.1) effectively converges towards the true solution $u(x)$ of (2.1), we find that

$$B_\infty^2 = \|\boldsymbol{K}f\|^2, \qquad D_\infty = ((\boldsymbol{I}-\boldsymbol{K})f, \boldsymbol{K}f), \qquad G_\infty = (f, \boldsymbol{K}f), \tag{4.11}$$

from which we can deduce $\alpha_\infty = \beta_\infty = 1$.

5. Numerical working of the solution for some equations of the second kind

The interesting cases are essentially those corresponding to $\|\boldsymbol{K}\| > 1$, for which we could not establish, unfortunately, accurate convergence criteria. For numerous examples, we have found empirically that the desired solution is generally obtained in fairly good conditions of computation when $\|\boldsymbol{K}\|$ is not greater than 3 or 4. As the nonlinear equation (3.1) has eventually several solutions (only one of these being a common solution for equation (2.1)), a stabilisation of (4.1) may occasionally exhibit worthless results. However, when such a tendency arises, this fact is immediately recognised as at least one of the two coefficients α_∞ and β_∞ will be different from unity.

In order to show completely how to operate with the new solving techniques which we have just advanced for Fredholm equations of the second kind, two relatively simple examples are detailed hereafter, with unidimensional variables x and y.

Example 1.
Let us solve

$$u(x) = 1 - 2\int_0^1 [1+3(x^2+y^2)]^{1/2}u(y)\,\mathrm{d}y. \tag{5.1}$$

Here the kernel

$$K(x, y) = K(y, x) = -2[1+3(x^2+y^2)]^{1/2} \tag{5.2}$$

reveals a Hilbert–Schmidt symmetry. Using Kellogg's method to evaluate the spectral norm (Krasnov *et al* 1977), we find

$$\|\boldsymbol{K}\| \approx 3.461, \tag{5.3}$$

i.e. a value much greater than one. Then, adjusting only β_n at every iteration step,

using formula (4.5), we have an absolute error Δu less than 1×10^{-7} everywhere on $[0, 1]$ for $n = 7$ (computation by Simpson's rule with an integration step equal to 0.05); simultaneously, $\beta_6 = 1 + 7\times10^{-9} \approx 1$. Values of $u_1(x)$ and $u_7(x) \approx u_\infty(x)$ are shown in table 1 and can be compared with $u_0(x) \equiv 1$.

Table 1.

x	$u_1(x)$	$u_7(x) \approx u_\infty(x)$
0.0	0.381 212	0.410 043
0.1	0.376 124	0.403 852
0.2	0.361 145	0.385 685
0.3	0.337 052	0.356 633
0.4	0.304 914	0.318 169
0.5	0.265 888	0.271 185
0.6	0.221 077	0.219 108
0.7	0.171 457	0.161 178
0.8	0.117 858	0.099 066
0.9	0.060 964	0.033 576
1.0	0.001 336	−0.034 652

Example 2.
On the interval $[-1, +1]$, let us now search for the solution which meets the boundary conditions

$$u(-1) = u(+1) = 0 \tag{5.4}$$

for the following second-order differential equation

$$u''(x) - 40xu(x) = 2. \tag{5.5}$$

As we know that

$$G(x, y) = -\tfrac{1}{2}\{|x-y| + xy - 1\} \tag{5.6}$$

is the Green function (Margenau and Murphy 1949) associated to $u''(x)$ and to conditions (5.4), we can easily write a Fredholm equation which is equivalent to (5.5) with the constraints (5.4). We have

$$u(x) = -x^2 + 1 + \int_{-1}^{+1} K(x, y)u(y)\,dy, \tag{5.7}$$

the introduced kernel being defined as

$$K(x, y) = 40yG(x, y). \tag{5.8}$$

As $K(x, y) \neq K(y, x)$ does not present Hilbert–Schmidt exchange symmetry, it is no longer possible to compute $\|\boldsymbol{K}\|$ using Kellogg's method. However, the definition (2.6) of the spectral norm allows us to state ($\forall u \in L_2$ and distinct from the null function)

$$\int_{-1}^{+1} [(\boldsymbol{K}u)(x)]^2\,dx \Big/ \int_{-1}^{+1} [u(x)]^2\,dx \leqslant \|\boldsymbol{K}\|^2. \tag{5.9}$$

With $u(x) \equiv 1$, we thus obtain

$$\|\boldsymbol{K}\| \geqslant \sqrt{172.698/2} = 9.292. \tag{5.10}$$

On the other hand, an upper bound of $\|\boldsymbol{K}\|$ is also available, using the inequality (Bass 1971)

$$\int_D \int_D [K(x, y)]^2 \, dx \, dy \geqslant \|\boldsymbol{K}\|^2 \Rightarrow \|\boldsymbol{K}\| \leqslant \sqrt{284.44} = 16.865. \tag{5.11}$$

We can conclude from (5.10) and (5.11) that the norm of $\boldsymbol{K}$ is rather high. Therefore, both parameters α_n and β_n were adjusted at every iterative step according to (4.10), Simpson's rule being used for all the necessary integrations with $\delta y = 0.05$. A stabilisation with $\Delta u < 2 \times 10^{-6}$ appears for $n = 23$ and we have the following results for the last parameters

$$\begin{aligned} &\alpha_{22} = 0.999\,999\,45, \qquad \beta_{22} = 0.999\,999\,59, \\ &\|u_{22} - f - \boldsymbol{K}u_{22}\|^2 = 1.77 \times 10^{-12}. \end{aligned} \tag{5.12}$$

Some values of $u_0(x)$, $u_1(x)$ and $u_\infty(x)$ are given in table 2.

Table 2.

x	$u_0(x) = -x^2 + 1$	$u_1(x)$	$u_{23}(x) \approx u_\infty(x)$
−1.0	0.000	0.000 000	0.000 000
−0.8	0.360	0.077 739	0.051 324
−0.6	0.640	0.111 125	0.084 022
−0.4	0.840	0.080 326	0.115 142
−0.2	0.960	−0.005 933	0.137 708
0.0	1.000	−0.119 886	0.122 838
0.2	0.960	−0.224 248	0.030 623
0.4	0.840	−0.281 734	−0.141 812
0.6	0.640	−0.264 579	−0.294 234
0.8	0.360	−0.164 057	−0.254 429
1.0	0.000	0.000 000	0.000 000

6. Examination of some essential features belonging to Fredholm equations of the first kind

A generalisation of the preceding techniques, partly iterative, partly variational, to the equations of the first kind, would be very interesting. However, before seeking efficient methods for this new problem, some characteristic features must be outlined for such Fredholm equations formulated as

$$(\boldsymbol{K}u)(x) = \int_D K(x, y) u(y) \, dy = f(x), \tag{6.1}$$

and for which the operator $\boldsymbol{K}$ continues to be restricted to finite norm. We need a suitable algorithm which is able to give the desired inversion

$$u(x) = (\boldsymbol{K}^{-1} f)(x), \tag{6.2}$$

but some fundamental remarks shall be written down before trying to build any computing scheme.

6.1. Our equation has no solution

This is in particular the case for the apparently simple equation

$$(\boldsymbol{K}u)(x) = \int_0^1 xyu(y)\,\mathrm{d}y = \mathrm{e}^x. \tag{6.3}$$

The displayed integral $(\boldsymbol{K}u)(x)$ is necessarily equal to a product of x by a constant; as $f(x) = \mathrm{e}^x$ does not present such a property, our problem is insolvable.

6.2. An infinite set of solutions may exist

Let us assume that some functions $v_1(x)$, $v_2(x)$, ... orthogonal to the kernel, i.e. such that

$$\int_D K(x, y)v_i(y)\,\mathrm{d}y = 0, \tag{6.4}$$

can be found. Then, when we already know a single solution $U_0(x)$ for (6.1), any other function defined as

$$u(x) = U_0(x) + \sum_i c_i v_i(x), \tag{6.5}$$

with arbitrary coefficients c_i, is also a valid solution for (6.1). When our integral transform changes $u(x)$ in $f(x) = (\boldsymbol{K}u)(x)$, *these particular kernels are inducing a loss of information*: as pointed out by (6.5), an indetermination characterised by an arbitrary linear combination of functions $v_i(x)$ will obviously remain when we try to rebuild $u(x)$.

Equations of that kind, with infinitely many possible solutions, are very frequent: for a convenient choice of $f(x)$, this will happen when the kernel $K(x, y)$ is degenerate or (for non-degenerate kernels) when the enumerable set of eigenfunctions is not complete.

6.3. Study of the integral transforms which, theoretically, do not induce a loss of information

Let

$$K(x, y) = \sum_{i=0}^{\infty} \mu_i \phi_i(x)\phi_i(y) \tag{6.6}$$

be the expansion of a symmetric Hilbert–Schmidt kernel with eigenvalues μ_i and with a complete set of orthogonal functions ϕ_i. In principle, the integral transform (6.1) should not induce any loss of information when $u(x)$ is changed into $f(x)$. To verify this statement, let us start from a function $u \in L_2$, such that

$$u(x) = \sum_i a_i \phi_i(x) \tag{6.7}$$

and giving

$$f(x) = \sum_i b_i \phi_i(x) = \sum_i \mu_i a_i \phi_i(x). \tag{6.8}$$

Conversely, a perfect knowledge of any function $f(x)$ will lead to the b_i values; we have

$$b_i = (f, \phi_i) = \int_D f(x)\phi_i(x)\,dx. \tag{6.9}$$

Afterwards, we are theoretically able to obtain the a_i coefficients. If the constraint

$$\sum_i \left(\frac{b_i}{\mu_i}\right)^2 < \infty \tag{6.10}$$

is effectively satisfied, the unique solution is formally the following

$$u(x) = \int_D \left[\sum_i \frac{\phi_i(x)\phi_i(y)}{\mu_i}\right] f(y)\,dy. \tag{6.11}$$

The inverse operator $\boldsymbol{K}^{-1}$ is thus associated with a formal kernel

$$K^{-1}(x, y) = \sum_i \frac{\phi_i(x)\phi_i(y)}{\mu_i} \tag{6.12}$$

and with the previous domain of integration D.

Now, for a non-degenerate kernel, the hypothesis

$$\int_D \int_D [K(x, y)]^2\,dx\,dy = \sum_i \mu_i^2 < \infty \tag{6.13}$$

indicates an accumulation of eigenvalues μ_i in the vicinity of zero. Many difficulties thence arise as $K^{-1}(x, y)$ cannot be square integrable. The formula (6.11) is mathematically exact but can only be applied when the μ_i's are perfectly accurate and when functions $f(x)$ and $\phi_i(x)$ are simultaneously analytical: in numerical analysis, $f(x)$ and $K(x, y)$ are only known through a finite number of sampling data with a limited number of digits and the problem is in fact completely different. For large indices i, when the eigenvalues μ_i are very close to zero, many terms $\mu_i a_i \phi_i(x)$ of (6.8) will be too small ($\forall x \in D$) to have any effect on the known digits of the available $f(x)$ data; it appears consequently that rather wide variations of these coefficients a_i may occur in $u(x)$ without leading to any visible change in its integral transform $f(x)$. Practically, this is equivalent to a loss of information which is very similar to that previously checked when functions orthogonal to the kernel do really exist: we may state that many small μ_i's behave numerically as if they were actually equal to exact zeros. This is why many authors point out that *Fredholm equations of the first kind belong to a class of ill posed problems* (Tikhonov and Arsenine 1976).

7. Solving possibilities for equations of the first kind

In order to solve (6.1), especially in the case of convolution problems, some authors start from the equivalent equation (Hill and Loup 1976)

$$u(x) = f(x) + [(\boldsymbol{I} - \boldsymbol{K})u](x), \tag{7.1}$$

and then apply the Liouville–Neumann iterative method with operator $(\boldsymbol{I} - \boldsymbol{K})$ instead of $\boldsymbol{K}$ previously used in formula (2.3). However, that kind of procedure will frequently fail, as a solution of (6.1) may only be obtained in such conditions when $f(x)$ has a

very special analytical behaviour. For instance, when $K(x, y)$ is a symmetric Hilbert–Schmidt kernel, the double inequality

$$1 \leq \|\boldsymbol{I} - \boldsymbol{K}\| \leq 1 + \|\boldsymbol{K}\| \tag{7.2}$$

is always satisfied: the lower bound is attained when the operator $\boldsymbol{K}$ is positive with $\|\boldsymbol{K}\| \leq 2$; on the contrary, the upper bound will be reached when the negative value $-\|\boldsymbol{K}\|$ is exactly one of the eigenvalues of $\boldsymbol{K}$.

Nevertheless, a convergence of the Liouville–Neumann sequence is effectively observed if

$$f(x) = \sum_{i \in I} b_i \phi_i(x), \tag{7.3}$$

the subset I of indices i being such that we have strictly

$$\forall i \in I, \qquad 0 < \mu_i < 2. \tag{7.4}$$

When conditions (7.4) are fulfilled, the absolute useful eigenvalues $|1 - \mu_i|$ of $(\boldsymbol{I} - \boldsymbol{K})$ will remain strictly less than unity; the iterative process is then convergent, as

$$\begin{aligned} u_0(x) &= \sum_{i \in I} b_i \phi_i(x), \\ u_1(x) &= \sum_{i \in I} b_i \phi_i(x)[1 + (1 - \mu_i)], \\ &\cdots \\ u_\infty(x) &= \sum_{i \in I} b_i \phi_i(x) \sum_{k=0}^{\infty} (1 - \mu_i)^k = \sum_{i \in I} \frac{b_i}{\mu_i} \phi_i(x). \end{aligned} \tag{7.5}$$

These preliminary remarks, related to (7.1), point out the great interest of positive operators for which the norm is actually lower than 2. Consequently, it seems preferable to write down, instead of (6.1),

$$(\boldsymbol{P}u)(x) = 2(\boldsymbol{K}^2 u)(x)/\|\boldsymbol{K}\|^2 = 2(\boldsymbol{K}f)(x)/\|\boldsymbol{K}\|^2 = g(x), \tag{7.6}$$

or, when $\boldsymbol{K}$ is already a positive operator,

$$(\boldsymbol{P}u)(x) = 2(\boldsymbol{K}u)(x)/\|\boldsymbol{K}\| = 2f(x)/\|\boldsymbol{K}\| = g(x). \tag{7.7}$$

Both equations (7.6) or (7.7) now contain a positive operator $\boldsymbol{P}$ with $\|\boldsymbol{P}\| = 2$, giving $\|\boldsymbol{I} - \boldsymbol{P}\| = 1$; simultaneously, the eigenvalues of $(\boldsymbol{I} - \boldsymbol{P})$ are widespread over the interval $[-1, +1]$.

With $\boldsymbol{P}$, the equation similar to (7.1), i.e.

$$u(x) = g(x) + [(\boldsymbol{I} - \boldsymbol{P})u](x), \tag{7.8}$$

will be easily worked out, using as nonlinear formula similar to (4.1) the expression

$$u_{n+1}(x) = \alpha_n g(x) + \beta_n [(\boldsymbol{I} - \boldsymbol{P})u](x), \tag{7.9}$$

in which α_n and β_n are deduced from the variational condition

$$\delta \|g - \boldsymbol{P}u_{n+1}\|^2 = 0. \tag{7.10}$$

The supplementary constraint $\alpha_n = 1$ $(\forall n)$ gives for β_n (compare with (4.5))

$$\beta_n = ((\boldsymbol{I} - \boldsymbol{P})g, (\boldsymbol{I} - \boldsymbol{P})\boldsymbol{P}u_n)/\|(\boldsymbol{I} - \boldsymbol{P})\boldsymbol{P}u_n\|^2. \tag{7.11}$$

When α_n and β_n are both adjusted at every step, formulae (4.10) are once more convenient, this time with the following values

$$A^2 = \|\boldsymbol{P}g\|^2, \qquad B_n^2 = \|(\boldsymbol{I}-\boldsymbol{P})\boldsymbol{P}u_n\|^2,$$
$$D_n = (\boldsymbol{P}g, (\boldsymbol{I}-\boldsymbol{P})\boldsymbol{P}u_n), \qquad F = (g, \boldsymbol{P}g), \tag{7.12}$$
$$G_n = (g, (\boldsymbol{I}-\boldsymbol{P})\boldsymbol{P}u_n).$$

Another advantage has to be noticed when our primary equation (6.1) is changed into (7.6): there always exists at least one solution for equation (7.6) as $g(x)$ and $(\boldsymbol{P}u)(x)$ are both necessarily linear combinations of $\boldsymbol{K}$-eigenfunctions.

7.1. Example involving a positive operator with weak singularity

We choose here a plane integration domain with two-dimensional variables x and y (coordinates x_1, x_2 for point M; y_1 and y_2 for M'): D will be a disc of unit radius with its centre in 0, to which we associate the kernel

$$K(M, M') = K(M', M) = 1/r_{MM'} \qquad (M, M' \in D); \tag{7.13}$$

$r_{MM'}$ is the distance between the two points M and M'.

We have now to examine the Fredholm equation of the first kind

$$f(M) = (\boldsymbol{K}u)(M) = \iint_D K(M, M')u(M')\,\mathrm{d}\sigma' \tag{7.14}$$

where $\mathrm{d}\sigma'$ denotes the elementary area around M'. From a physical point of view, it is very interesting to solve (7.14) as this typical equation can be set up easily when studying the transaxial tomography problem: the data $f(M)$ are experimentally obtained and the unknown $u(M)$ characterises the image intensity (Guy *et al* 1979a).

Here, the kernel (7.13) is no longer square integrable but it remains a kernel with a weak singularity, in Mikhlin's sense (1960). Such an operator has a reasonable behaviour with a norm always defined by (2.6). We obtain, using Kellogg's method,

$$\|\boldsymbol{K}\| \approx 5.438. \tag{7.15}$$

In order to test our solving technique, a cylindrical symmetric function (consequently symbolised by $f(r)$, r being the polar radius of M) has been firstly worked out from the equally cylindrical symmetric function

$$u(r) = (1 - \tfrac{1}{2}r^2)^{1/2}, \tag{7.16}$$

using (7.14) as integral transform.

As operator $\boldsymbol{K}$ is positive and 'does not produce theoretically any loss of information', it is sufficient to employ equation (7.7). Moreover, it is also possible to utilise Aitken's procedure for convergence acceleration (Brezinski 1978); with

$$u_\infty(r) \approx u_4(r) - \frac{[u_6(r) - u_5(r)][u_5(r) - u_4(r)]}{u_4(r) - 2u_5(r) + u_6(r)}, \tag{7.17}$$

we have thus found that the exact starting function $u_e(r)$ is practically recovered, the absolute error being less than 2×10^{-3} everywhere over the whole disc D. Some results of the numerical computation are collected in table 3 below.

Table 3.

r	u_e	$u_0 = f_0/2.719$	u_1	u_2	u_∞
0.0	1.0000	2.1004	0.4847	0.9843	0.9997
0.1	0.9975	2.0930	0.4841	0.9828	0.9972
0.2	0.9899	2.0711	0.4825	0.9768	0.9896
0.3	0.9772	2.0344	0.4794	0.9671	0.9768
0.4	0.9592	1.9626	0.4745	0.9531	0.9586
0.5	0.9354	1.9152	0.4673	0.9341	0.9348
0.6	0.9055	1.8315	0.4568	0.9093	0.9050
0.7	0.8689	1.7301	0.4420	0.8770	0.8688
0.8	0.8246	1.6082	0.4211	0.8347	0.8258
0.9	0.7714	1.4590	0.3906	0.7760	0.7741
1.0	0.7071	1.2437	0.3341	0.6665	0.7058

From this last example, we can assert that the form (7.8) for a Fredholm equation of the first kind is very helpful: using judiciously the constraint (7.10), we are thus given an efficient iterative process leading to the desired solution with a good accuracy.

We finally conclude that we have here detailed, in the case of Fredholm integral equations, some interesting optimisation iterative methods. These are very competitive with other ones sharing the same main ideas (i.e. adjusting introduced parameters at every step of iteration), as the gradient techniques which are often utilised when the operators are matrices (Faddeev and Faddeeva 1963, Fletcher 1980, Ciarlet 1982).

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